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Soliton-like lamb waves☆

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ABSTRACT

The velocity and polarization of acoustic Lamb waves, propagating in the directions of elastic symmetry of single-layer and double-layer anisotropic media at vanishingly low frequencies (soliton-like waves), are investigated. The method of fundamental matrices is used to construct solutions. The conditions for soliton-like Lamb waves to exist are analysed.

1. Introduction

Solitons, or propagating waves,[1] are isolated waves, recalling a shock wave front, and satisfy the following conditions (Conditions A): they propagate with constant velocity without appreciable attenuation and do not interact with other waves.[2] In hydrodynamics, such waves are described by the non-linear Korteweg–de Vries equation.[3–5]

In this paper we analyse soliton-like waves propagating in an elastic layer and which satisfy Conditions A. Unlike solitons in hydrodynamics they are described by the solution of a system of *linear* equations, known as the Christoffel equations for Lamb waves at an angular frequency approaching zero or a wave number approaching zero. Linear soliton-like waves in plates were detected for the first time[6] in the course of model numerical experiments. These waves have been investigated numerically in circular cylindrical rods[7–10] as special solutions of the linear Pochhammer–Chree equation and were considered within the framework of non-linear equations similar to the Korteweg–de Vries equation.[11–13]

Numerical results[6–10] confirm that for the lowest mode, Lamb waves (or Pochhammer–Chree waves) for a small wave number r, the phase velocity $c(r)$ satisfies the condition

$$c(r) = O(r^n), \quad r \to 0 \tag{1.1}$$

where $n > 0$ is a certain integer. Note that the exponent n could not be determined in the numerical experiments. Below, a somewhat more accurate condition (see the note at the end of Section 3) will be used to obtain the phase velocity of a soliton-like wave.

By considering a surface wave $\mathbf{u}(\mathbf{x}, \omega, t)$, represented in the form of the superposition of two harmonic waves

$$\mathbf{u}(\mathbf{x}, \omega, t) = \mathbf{u}_0(\mathbf{x}, \omega)e^{i\omega t} - \mathbf{u}_0(\mathbf{x}, \omega)e^{i\omega(t-t_0)} = \mathbf{u}_0(\mathbf{x}, \omega)(e^{i\omega t} - e^{i\omega(t-t_0)})$$

propagating with a phase shift $e^{i(\pi+\omega t_0)}$, it can be seen that the resulting wave, when $\omega \to 0$ and $t_0 = 1/\omega$, is a solitary wave – a shock front. By choosing superpositions with different harmonic waves one can obtain other forms of solitary waves. Conditions A are satisfied in a trivial way for any waves which are the solution of systems of *linear* differential equations.

The fact that soliton-like Lamb waves in plates propagate at vanishingly low frequencies explains the low energy required to excite them. In fact, the distributed kinetic energy of a Lamb wave is described by the expression

$$E_{\text{kin}} \equiv \frac{1}{2}\rho|\dot{\mathbf{u}}|^2 = \frac{1}{2}\rho|\mathbf{m}|^2\omega^2 \tag{1.2}$$

where $\mathbf{m}$ is the amplitude of the displacements, which depends on the position of the point inside the plate. Hence, for a limited amplitude and when $\omega \to 0$ the kinetic energy approaches zero. It can be shown that the distributed potential energy is also proportional to the square of the amplitude and the frequency.

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In addition to Rayleigh waves, Lamb waves[14] play an important role in transferring energy and are very often employed for nondestructive testing. The displacement field of a wave propagating in an *isotropic* layer, is usually represented in the form

$$\mathbf{u}(\mathbf{x},t) = \left(\sum_{p=1}^{4} \mathbf{m}_p C_p e^{ir\gamma_p x'}\right) e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{1.3}$$

where $\mathbf{u}$ are the displacements in the layer and $\mathbf{m}_p$ are unit amplitudes (polarization vectors). It is assumed that each of the vectors $\mathbf{m}_p$ lies in the sagittal plane (it is defined by the vector $\mathbf{n}$, which specifies the direction of the wave front propagation, and the unit vector ν, normal to the middle plane of the layer), $x' \equiv \nu\cdot\mathbf{x}$ is the coordinate along the vector ν, and the parameters γ_p are found from the Christoffel equation derived later. In representation (1.3) the waves

$$\mathbf{u}^p(\mathbf{x},t) = \mathbf{m}_p e^{ir\gamma_p x'} e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{1.4}$$

are usually called partial waves. In expressions (1.3) the unknown (complex) coefficients C_p are found, apart from a factor, from the boundary conditions on the free boundary surfaces

$$x' = \pm h:\ \mathbf{t}_\nu \equiv \boldsymbol{\nu}\cdot\mathbf{C}\cdot\cdot\nabla_\mathbf{x}\mathbf{u} = 0 \tag{1.5}$$

where $\mathbf{C}$ is the elasticity tensor and $2h$ is the layer thickness. The exponential factor $e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}$ in Eq. (1.3) corresponds to the propagation of a plane wave front $\mathbf{n}\cdot\mathbf{x}=\text{const}$.

Representation (1.3) also arises in the case of anisotropic layers, if the following conditions are satisfied (Conditions B): the elasticity tensor possesses an axis of elastic symmetry and the wave propagates in the direction of this axis. The first of these conditions is equivalent to the presence in the elasticity tensor of a monoclinic symmetry group; in this case the elasticity tensor contains 13 independent separable components. When Conditions B break down the amplitudes of the partial waves may not lie in the sagittal plane, which leads to the need to take into account six partial waves in representation (1.3) for the Lamb wave.[15]

In the case of a multilayer medium, consisting of two or more contacting layers, the corresponding solutions are usually obtained by two different methods. These methods are known as the transfer matrix method (sometimes called the Thomson–Haskell method after its developers[16,17]) and the global matrix method.[18,19] The transfer matrix method is based on a successive solution of contact boundary-value problems on the interfaces and the construction of corresponding transfer matrices. This method will be discussed in more detail below. The global matrix method involves solving ordinary differential equations with piecewise-homogeneous coefficients, leading finally to the construction of a special "global" matrix.

Below we develop a version of the transfer matrix method based on the construction of fundamental matrices which enable soliton-like Lamb waves in media with arbitrary anisotropy to be investigated analytically.

2. Fundamental relations

We will assume below that all the layers are homogeneous and linearly hyperelastic. The equations of motion for a homogeneous isotropic elastic medium can be represented in the form

$$\mathbf{A}(\partial_x,\partial_t)\mathbf{u} \equiv \operatorname{div}_x\mathbf{C}\cdot\cdot\nabla_x\mathbf{u} - \rho\ddot{\mathbf{u}} = 0 \tag{2.1}$$

The elasticity tensor C is assumed to be positive definite

$$(\mathbf{A}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{A}) \equiv \sum_{i,j,m,n} A_{ij}C^{ijmn}A_{mn} > 0, \qquad \underset{\mathbf{A}\in\operatorname{sym}(R^3\otimes R^3),\ \mathbf{A}\neq 0}{\forall\mathbf{A}} \qquad \operatorname{sym}\mathbf{A} = \frac{1}{2}(\mathbf{A}+\mathbf{A}^t) \tag{2.2}$$

Note that, in the case of an isotropic elastic medium, condition (2.2) for the tensor to be positive definite is equivalent to the conditions

$$\mu > 0, \quad \lambda > -2\mu/3 \tag{2.3}$$

where λ and μ are Lamé constants.

Following the method described previously,[15,20] we will consider a representation for the Lamb wave propagating in a layer with arbitrary elastic anisotropy, which is more general than (1.3),

$$\mathbf{u}(\mathbf{x},t) \equiv \mathbf{f}(x'')e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}$$

where $x'' = irx'$ is a dimensionless variable and $\mathbf{f}$ is an unknown vector function, which defines the change in amplitude on the wave front. Substituting this expression into Eq. (2.1), we obtain an ordinary differential equation in the function $\mathbf{f}$ (known as the Christoffel equation for the Lamb wave)

$$-r^2(\mathbf{A}_1\partial_{x''}^2 + \mathbf{A}_2\partial_{x''} + \mathbf{A}_3)\cdot\mathbf{f} = 0 \tag{2.4}$$

where

$$\mathbf{A}_1 = \boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}, \quad \mathbf{A}_2 = \boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}, \quad \mathbf{A}_3 = \mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I} \tag{2.5}$$

For subsequent analysis we will reduce Eq. (2.4) to a first-order ordinary matrix differential equation, by introducing the auxiliary function $\mathbf{w} = \partial_{x''}\mathbf{f}$. We obtain

$$\partial_{x''}\begin{Vmatrix}\mathbf{f}\\ \mathbf{w}\end{Vmatrix} = \mathbf{G}\cdot\begin{Vmatrix}\mathbf{f}\\ \mathbf{w}\end{Vmatrix}; \quad \mathbf{G} = \begin{Vmatrix}\mathbf{O} & \mathbf{I}\\ -\mathbf{A}_1^{-1}\cdot\mathbf{A}_3 & -\mathbf{A}_1^{-1}\cdot\mathbf{A}_2\end{Vmatrix}, \quad \det\mathbf{G} = \frac{\det\mathbf{A}_3}{\det\mathbf{A}_1} \tag{2.6}$$

Here **G** is a sixth-order matrix for a medium with arbitrary anisotropy and a fourth-order matrix for the case when Conditions B are satisfied, while **O** and **I** are zero and unit 3 × 3 matrices. The matrix **G** enables us to represent the general solution of Eq. (2.6) in the form

$$\begin{Vmatrix}\mathbf{f}\\ \mathbf{w}\end{Vmatrix}_0 = e^{ir\mathbf{G}x'}\cdot\mathbf{U}$$

where **U** is a six-dimensional, generally speaking, complex vector, defined, apart from a scalar factor, by boundary conditions (1.5). We then obtain representation (2.3) in the form

$$\begin{Vmatrix}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\upsilon}(\mathbf{x},t)\end{Vmatrix} = (e^{ir\mathbf{G}x'}\cdot\mathbf{U})e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}; \quad \boldsymbol{\upsilon}(\mathbf{x},t) = \mathbf{w}(x'')e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{2.7}$$

Note that representation (2.7) also remains true in the case of non-semisimple degeneration of the matrix **G**, i.e., when there are Jordan blocks in the canonical normal form of the matrix **G**.

3. A soliton-like wave in a homogeneous anisotropic layer

Substituting the solution in the form (2.7) into boundary conditions (1.5) we obtain

$$\mathbf{M}\cdot\mathbf{U} = 0; \quad \mathbf{M} = \begin{Vmatrix}(\mathbf{A}_4,\mathbf{A}_1)\cdot e^{+ir\mathbf{G}h}\\ -(\mathbf{A}_4,\mathbf{A}_1)\cdot e^{-ir\mathbf{G}h}\end{Vmatrix}, \quad \mathbf{A}_4 = \mathbf{v}\cdot\mathbf{C}\cdot\mathbf{n} \tag{3.1}$$

The presence of a non-trivial solution of Eq. (3.2) is equivalent to the condition

$$\det\mathbf{M} = 0 \tag{3.2}$$

which is known as the dispersion equation for a Lamb wave, since this equation defines the phase velocity as an implicit function of the frequency or the wave number.

For $r = 0$ and any anisotropy of the elastic homogeneous layer, Eq. (3.2) is satisfied identically, which follows from expression (3.1) for the matrix **M** when $r = 0$. However, the solution obtained for $r = 0$ is empty: it does not ensure that condition (3.2) is satisfied for small r and it does not determine the phase velocity of a soliton-like wave. We will use condition (1.1) below to find this wave. Taking expression (3.2) into account, condition (1.1) can be written in the form of the following conditions

$$\frac{d^k}{dr^k}c(r) \equiv -\left.\frac{\partial_r^k\det\mathbf{M}}{\partial_c\det\mathbf{M}}\right|_{r=0} = 0, \quad k = 1,\dots,n \tag{3.3}$$

It can be seen that conditions (3.3) are equivalent to the equalities

$$\left.\partial_r^k\det\mathbf{M}\right|_{r=0} = 0, \quad k = 1,\dots,n \tag{3.4}$$

We will expand the exponential representations occurring in (3.1) in a Taylor series. This gives the matrix **M** in the following form

$$\mathbf{M} = \begin{Vmatrix}\mathbf{A}_4 & \mathbf{A}_1\\ -\mathbf{A}_4 & \mathbf{A}_1\end{Vmatrix} + \frac{irh}{1!}\begin{Vmatrix}-\mathbf{A}_3 & \mathbf{A}_4 & -\mathbf{A}_2\\ -\mathbf{A}_3 & \mathbf{A}_4 & -\mathbf{A}_2\end{Vmatrix} + \frac{(irh)^2}{2!}\begin{Vmatrix}\mathbf{B}_1 & \mathbf{B}_2\\ -\mathbf{B}_1 & -\mathbf{B}_2\end{Vmatrix} + \frac{(irh)^3}{3!}\begin{Vmatrix}\mathbf{B}_3 & \mathbf{B}_4\\ \mathbf{B}_3 & \mathbf{B}_4\end{Vmatrix} + O(r^4) \tag{3.5}$$

$$\mathbf{B}_1 = -\mathbf{A}_4\mathbf{A}_1^{-1}\mathbf{A}_3 + \mathbf{A}_2\mathbf{A}_1^{-1}\mathbf{A}_3, \quad \mathbf{B}_2 = -\mathbf{A}_4\mathbf{A}_1^{-1}\mathbf{A}_2 - \mathbf{A}_3 + \mathbf{A}_2\mathbf{A}_1^{-1}\mathbf{A}_2$$

$$\mathbf{B}_3 = \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{A}_1^{-1}\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_3 - (\mathbf{A}_2\mathbf{A}_1^{-1})^2\mathbf{A}_3$$

$$\mathbf{B}_4 = \mathbf{B}_1 + \mathbf{A}_4(\mathbf{A}_2\mathbf{A}_1^{-1})^2 + \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2 - (\mathbf{A}_2\mathbf{A}_1^{-1})^2\mathbf{A}_2$$

Confining ourselves, for small r, to the first four terms of the Taylor expansion in (3.5) and using Schur's formula,[21] we obtain conditions (3.4) in the form

$$\partial_r^k \det\mathbf{M}|_{r=0} \equiv \partial_r^k(\det\mathbf{X}_+\det(\mathbf{Y}_- - \mathbf{X}_-\mathbf{X}_+^{-1}\mathbf{Y}_+))\big|_{r=0} = 0, \quad k = 1, \dots, n$$

$$\mathbf{X}_\pm = \pm\mathbf{A}_4 - (irh)\mathbf{A}_3 \pm \frac{(irh)^2}{2}\mathbf{B}_1 + \frac{(irh)^3}{3!}\mathbf{B}_3$$

$$\mathbf{Y}_\pm = \pm\mathbf{A}_1 + (irh)(\mathbf{A}_4 - \mathbf{A}_2) \pm \frac{(irh)^2}{2}\mathbf{B}_2 + \frac{(irh)^3}{3!}\mathbf{B}_4 \tag{3.6}$$

The matrices occurring in Eq. (3.6) are correctly defined if the phase velocity c differs from the velocity of body waves, propagating in the direction of the wave normal $\mathbf{n}$ to the wave front. This condition will be assumed to be satisfied below. Equations (3.6) give the necessary and sufficient conditions for a soliton-like wave, satisfying condition (1.1), to exist.

The presence in conditions (3.3) and (3.6) of the parameter $n \geq 1$, which depends on the anisotropy and characterizes the decrease in the phase velocity $c(r)$ as $r \to 0$, is due to the fact that, for small r, we can represent the determinant of the matrix $\mathbf{M}$ in the form

$$\det\mathbf{M} = r^n V_n + o(r^n), \quad r \to 0 \tag{3.7}$$

where V_n is a certain constant, independent of r. Bearing (3.7) in mind, it becomes clear that conditions (3.3) and (3.6) define the phase velocity for which the lowest coefficient V_n of the asymptotic expansion (as $r \to 0$) of the determinant of the matrix $\mathbf{M}$ is degenerate.

4. A soliton-like wave in a homogeneous isotropic layer

For an isotropic elastic layer we have

$$\mathbf{A}_1 = (\lambda + 2\mu)\nu\otimes\nu + \mu(\mathbf{n}\otimes\mathbf{n} + \mathbf{w}\otimes\mathbf{w}), \quad \mathbf{A}_2 = (\lambda+\mu)(\boldsymbol{\nu}\otimes\mathbf{n} + \mathbf{n}\otimes\boldsymbol{\nu})$$

$$\mathbf{A}_3 = (\lambda + 2\mu - \rho c^2)\mathbf{n}\otimes\mathbf{n} + (\mu - \rho c^2)(\nu\otimes\nu + \mathbf{w}\otimes\mathbf{w}), \quad \mathbf{A}_4 = \lambda\boldsymbol{\nu}\otimes\mathbf{n} + \mu\mathbf{n}\otimes\boldsymbol{\nu}$$

$$\mathbf{w} = \boldsymbol{\nu}\times\mathbf{n} \tag{4.1}$$

From expressions (4.1) and (2.6) we obtain the matrix $\mathbf{G}$ in the form

$$\mathbf{G} = \left\|\begin{matrix}\mathbf{O} & \mathbf{I}\\ \mathbf{G}_1 & \mathbf{G}_2\end{matrix}\right\|, \quad \mathbf{G}_1 = \mathrm{diag}\{a^2d^{-1}, b^2d, a^2\}, \quad \mathbf{G}_2 = \left\|\begin{matrix}0 & -1+d^1 & 0\\ 1-d & 0 & 0\\ 0 & 0 & 0\end{matrix}\right\|$$

$$a = \left(\frac{\rho c^2}{\mu} - 1\right)^{1/2}, \quad b = \left(\frac{\rho c^2}{\lambda + 2\mu} - 1\right)^{1/2}, \quad d = \frac{\lambda + 2\mu}{\mu} \tag{4.2}$$

For an isotropic medium the fundamental matrix $e^{ir\mathbf{G}x'}$ can be constructed explicitly. To do this we reduce the matrix $\mathbf{G}$ to the Jordan normal form

$$\mathbf{G} = \mathbf{W}\cdot\mathbf{D}\cdot\mathbf{W}^{-1} \tag{4.3}$$

where $\mathbf{W}$ is a matrix consisting of the right (un-normalised) eigenvectors $\mathbf{G}$, arranged in columns

$$\mathbf{W} = \left\|\begin{matrix}1 & 1 & 1 & 1 & 0 & 0\\ -a & a & b^{-1} & -b^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & a^{-1} & -a^{-1}\\ a & -a & b & -b & 0 & 0\\ -a^2 & -a^2 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\end{matrix}\right\| \tag{4.4}$$

and $\mathbf{D}$ is a diagonal matrix

$$\mathbf{D} = \mathrm{diag}\{a, -a, b, -b, a, -a\} \tag{4.5}$$

It can be proved that, for any permissible values of λ and μ, satisfying condition (2.3), matrix (4.2) is semisimple.

Taking equalities (4.3)–(4.6) into account, we can represent the fundamental matrix in the form

$$e^{irx'\mathbf{G}} = \mathbf{W}\cdot e^{irx'\mathbf{D}}\cdot\mathbf{W}^{-1} \tag{4.6}$$

Combining relations (3.2), (4.1), (4.2) and (4.7) and expanding the exponential matrix $e^{irx'\mathbf{D}}$ in a Taylor series, we obtain the matrix **M** in the form

$$\mathbf{M} = \begin{Vmatrix} 0 & \lambda & 0 & \lambda+2\mu & 0 & 0 \\ \mu & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \\ 0 & -\lambda & 0 & -(\lambda+2\mu) & 0 & 0 \\ -\mu & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu \end{Vmatrix} + irh \begin{Vmatrix} a^2\mu & 0 & 0 & 0 & -\mu & 0 \\ 0 & b^2(\lambda+2\mu) & 0 & -\lambda & 0 & 0 \\ 0 & 0 & a^2\mu & 0 & 0 & 0 \\ a^2\mu & 0 & 0 & 0 & -\mu & 0 \\ 0 & b^2(\lambda+2\mu) & 0 & -\lambda & 0 & 0 \\ 0 & 0 & a^2\mu & 0 & 0 & 0 \end{Vmatrix} +$$

$$+\frac{(irh)^2}{2} \begin{Vmatrix} -a^2d^{-1}\lambda & 0 & 0 & \lambda+\rho c^2 & 0 & 0 \\ 0 & -b^2(\lambda+2\mu) & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & a^2\mu \\ a^2d^{-1}\lambda & 0 & 0 & -(\lambda+\rho c^2) & 0 & 0 \\ 0 & b^2(\lambda+2\mu) & 0 & 0 & -f & 0 \\ 0 & 0 & 0 & 0 & 0 & -a^2\mu \end{Vmatrix} + o(r^2), \quad r \to 0$$

$$f = \frac{\rho c^2(\lambda+2\mu) - \mu(3\lambda+4\mu)}{\lambda+2\mu} \tag{4.7}$$

Dispersion equation (3.4) then gives the following value of the velocity of the soliton-like Lamb wave, propagating in an isotropic layer

$$c_s = 2\sqrt{\frac{\mu(\lambda+\mu)}{\rho(\lambda+2\mu)}} \tag{4.8}$$

which is independent of the plate thickness. An analysis of (4.8) shows that for any permissible values of the parameters λ and μ this velocity lies in the range $c_T^{\text{bulk}} < c_s \le c_L^{\text{bulk}}$, where c_T^{bulk} and c_L^{bulk} are the velocities of the bulk transverse and longitudinal waves respectively. Only when $\lambda = 0$ does the velocity of the soliton-like wave become equal to the velocity of the bulk longitudinal wave.

Using eigenvectors (4.4) it becomes possible to determine the polarization of a soliton-like wave. Substituting the velocity c_s into Eq. (4.8), we obtain (apart from a constant) the eigenvector **U**, corresponding to the zero eigenvalue of the matrix **M**

$$\mathbf{U} = \left\{\frac{3\lambda+\mu}{-\lambda+2\mu}, \frac{3\lambda+\mu}{-\lambda+2\mu}, 1, 1, 0, 0\right\} \tag{4.9}$$

Multiplying the polarization vectors of the partial waves (4.4) by vector (4.9), we obtain that, when $r = 0$, a soliton-like Lamb wave has a polarization directed along the vector ν and an amplitude that is constant throughout the plate thickness. Hence, in the limiting case when $r = 0$, a soliton-like wave propagating with a velocity (4.8) is a surface SV-wave (a transverse wave with vertical polarization).

5. A soliton-like wave in a double-layer anisotropic plate

Consider a double-layer plate, consisting of two homogeneous elastic anisotropic layers of thickness $2h_1$ and $2h_2$, at the interface of which the conditions of ideal mechanical contact are formulated, i.e.

$$\{\mathbf{u}(-h_1) = \mathbf{u}(+h_2), \mathbf{t}_{-\mathbf{v}}(-h_1) = -\mathbf{t}_{\mathbf{v}}(+h_2) \tag{5.1}$$

while the external surfaces of the plate are assumed to be force-free

$$\{\mathbf{t}_{\mathbf{v}}(+h_1) = 0, \quad \mathbf{t}_{-\mathbf{v}}(-h_2) = 0 \tag{5.2}$$

By analogy with representation (2.7), we will represent the six-dimensional field in each of the layers in the form of corresponding fundamental matrices $e^{ir\mathbf{G}_k x'}$

$$\begin{Vmatrix} \mathbf{u}_k(\mathbf{x},t) \\ \boldsymbol{\upsilon}_k(\mathbf{x},t) \end{Vmatrix} = (e^{ir\mathbf{G}_k x'} \cdot \mathbf{U}_k) e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)}, \quad k = 1, 2 \tag{5.3}$$

(the subscript k relates to the corresponding layer). Substituting representation (5.3) into the conditions on the interface (5.1) we obtain

$$\mathbf{F}_1 \cdot (e^{-ir\mathbf{G}_1 h_1} \cdot \mathbf{U}_1) = \mathbf{F}_2 \cdot (e^{+ir\mathbf{G}_2 h_2} \cdot \mathbf{U}_2)$$

$$\mathbf{F}_k = \left\| \begin{matrix} \mathbf{I} & \mathbf{O} \\ (\mathbf{A}_4)_k & (\mathbf{A}_1)_k \end{matrix} \right\|, \quad k = 1, 2 \tag{5.4}$$

It can be seen that when the condition for positive definiteness (2.2) is satisfied for the elasticity tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ all the 6×6 matrices (5.4) are non-degenerate. This enables us to express the six-dimensional vector $\mathbf{U}_2$ in terms of the vector $\mathbf{U}_1$

$$\mathbf{U}_2 = e^{-ir\mathbf{G}_2 h_2} \cdot \mathbf{F}_2^{-1} \cdot \mathbf{F}_1 \cdot e^{-ir\mathbf{G}_1 h_1} \cdot \mathbf{U}_1 \tag{5.5}$$

Taking (5.5) into account, boundary conditions (5.2) can be written in the form

$$\mathbf{M} \cdot \mathbf{U}_1 = 0 \tag{5.6}$$

where

$$\mathbf{M} = \left\| \begin{matrix} ((\mathbf{A}_4)_1, (\mathbf{A}_1)_1) \cdot e^{+ir\mathbf{G}_1 h_1} \\ ((-\mathbf{A}_4)_2, (-\mathbf{A}_1)_2) \cdot e^{-ir\mathbf{G}_2 h_2} \cdot \mathbf{F}_2^{-1} \cdot \mathbf{F}_1 \cdot e^{-ir\mathbf{G}_1 h_1} \end{matrix} \right\| \tag{5.7}$$

The existence of non-trivial solutions of Eq. (5.6) is equivalent to the satisfaction of condition (3.2), while the condition for a soliton-like wave to exist, in addition to (3.2), will be the satisfaction of Eq. (3.3).

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